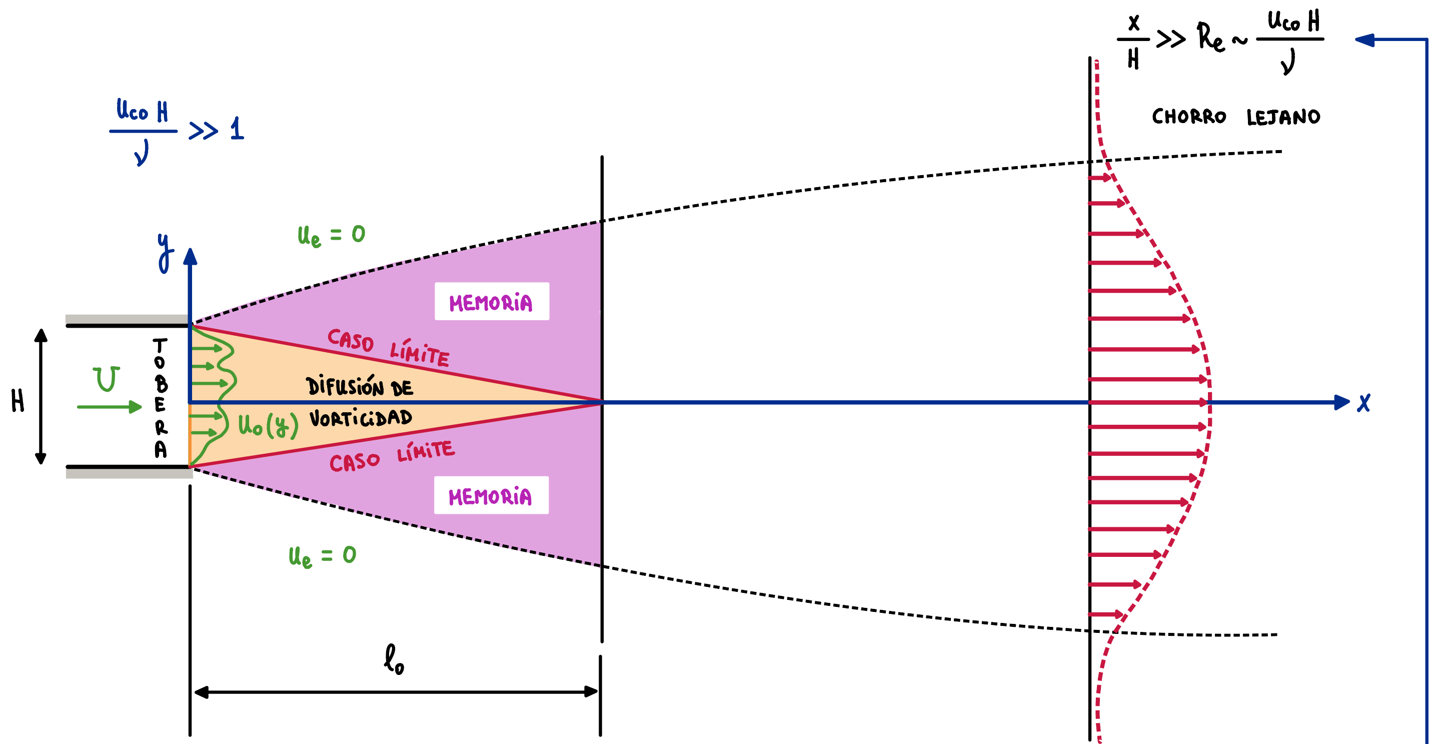
Chorro Plano Laminar

$\frac{U_{co} H}{\nu} \gg 1$



EFECTOS VISCOSOS: encargado de la "pérdida de memoria" de la tobera de salida.

$$\text{ECdM}_x: \quad \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\sim U_{co}\frac{U_{co}}{\ell_0}} = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + U_m\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \underbrace{\frac{\partial^2 u}{\partial y^2}}_{\sim \nu \frac{U_{co}}{H^2}}\right)$$

Igualando órdenes de magnitud de los términos CONVECTIVO y VISCOSO:

$$U_{co}\frac{U_{co}}{\ell_0} \sim \nu\frac{U_{co}}{H^2} \longrightarrow \frac{\ell_0}{H} \sim \frac{U_{co} H}{\nu} \sim Re \gg 1$$

$$\boxed{\frac{U_{co} H}{\nu} \sim Re \gg 1}$$

$$\boxed{\ell_0 \sim H\, Re}$$

En $x \gg \ell_0$: $\frac{x}{\ell_0} \gg 1 \longrightarrow \frac{x}{H\,Re} \gg 1 \longrightarrow$ $\boxed{\frac{x}{H} \gg Re}$ (LA MEMORIA DE LA VELOCIDAD EN $x = 0$ DESAPARECE)

Ecuaciones del flujo:

$$\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}_{\text{1 c.c. de } v \text{ en } y} = 0 \qquad \underbrace{u\frac{\partial u}{\partial x}}_{\text{1 c.c. de } u \text{ en } x} + v\frac{\partial u}{\partial y} = \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\text{2 c.c. de } u \text{ en } y}$$

Ambas ecuaciones son también válidas cerca de la tobera pues la respuesta viscosa principal es en "y"

Condiciones de contorno:

- Simetría: $\begin{cases} \blacksquare\ v(y) = v(-y) \longrightarrow y=0:\ v=0 \\ \blacksquare\ u(y) = u(-y) \longrightarrow y=0:\ \Delta_y u = 0 \longrightarrow y=0:\ \dfrac{\partial u}{\partial y}=0 \end{cases}$
- $y \rightarrow \infty:\ u \rightarrow u_e = 0$
- Falta la c.c. en "x" para "u" $\longrightarrow$ Imponemos que en una estación $x^*/x^* \gg H\,Re$

  Conocemos u: $\boxed{u(x^*,y) = u^*(y)}$ ES PARTE DE LA SOLUCIÓN $\longrightarrow$ DIFICULTAD

Si escribimos la $ECdM_x$ en forma conservativa e integramos en la dirección transversal:

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(u\cdot v) = \frac{\partial}{\partial y}\left(\nu\,\frac{\partial u}{\partial y}\right)$$

$$\int_0^\infty \frac{\partial}{\partial x}(u^2)\,dy + \int_0^\infty \frac{\partial}{\partial y}(u\cdot v)\,dy = \int_0^\infty \frac{\partial}{\partial y}\left(\nu\,\frac{\partial u}{\partial y}\right)dy$$

$$\frac{d}{dx}\int_0^\infty u^2 dy + \underbrace{u\cdot v\Big|_0^\infty}_{0} = \underbrace{\nu\,\frac{\partial u}{\partial y}\Big|_0^\infty}_{0}$$

$$\frac{d}{dx}\int_0^\infty u^2 dy = 0$$

$$\int_0^\infty u^2 dy = I = cte \quad (\forall\, x \in \text{chorro lejano})$$

Para que valga $\forall x$ integramos mejor entre $-\infty$ y $+\infty$:

$$\frac{d}{dx}\int_{-\infty}^\infty u^2 dy + \underbrace{u\cdot v\Big|_{-\infty}^\infty}_{0} = \underbrace{\nu\,\frac{\partial u}{\partial y}\Big|_{-\infty}^\infty}_{0}$$

$$\frac{d}{dx}\int_{-\infty}^\infty u^2 dy = 0$$

INVARIANTE $\forall x \longrightarrow$ VOY A $x=0$

$$\int_{-\infty}^\infty u^2 dy = 2I = cte = \int_{-H/2}^{H/2} u_0^2(y)\,dy$$

$u(y) = U$

$$\boxed{I = \frac{1}{2}\int_{-H/2}^{H/2} u_0^2(y)\,dy = \frac{U^2 H}{2}}$$

$$x \sim H\,Re \sim \frac{U^2 H}{\nu}$$

El chorro tiene un núcleo central no viscoso, limitado por una capa de cortadura que va incorporando líquido del exterior al chorro.

NO HAY SOLUCIÓN DE SEMEJANZA pues U y H intervienen por separado.

$$x \gg \frac{U^2 H}{\nu}$$

Los efectos viscosos afectan ya a todo el chorro y se pierde el detalle de lo que ocurre cerca de la tobera.

SOLUCIÓN DEL CAMPO LEJANO: interviene sólo I, no U y H por separado.

Eliminamos el parámetro ν de la ECM$_x$: $x, \dfrac{y}{\sqrt{\nu}}, u, \dfrac{v}{\sqrt{\nu}}$

$$\frac{\partial u}{\partial x} + \frac{\partial\left(\frac{v}{\sqrt{\nu}}\right)}{\partial\left(\frac{y}{\sqrt{\nu}}\right)} = 0$$

$$u\frac{\partial u}{\partial x} + \frac{v}{\sqrt{\nu}}\frac{\partial u}{\partial\left(\frac{y}{\sqrt{\nu}}\right)} = \frac{\partial^2 u}{\partial\left(\frac{y}{\sqrt{\nu}}\right)^2}$$

$$\frac{y}{\sqrt{\nu}} = 0: \quad v = 0\ ; \quad \frac{\partial u}{\partial\left(\frac{y}{\sqrt{\nu}}\right)} = 0$$

$$\frac{y}{\sqrt{\nu}} \to \infty: \quad u \to 0$$

$$\forall x: \int_0^\infty u^2\, d\left(\frac{y}{\sqrt{\nu}}\right) = \frac{I}{\sqrt{\nu}}$$

VAR. INDEPENDIENTES: $x, \dfrac{y}{\sqrt{\nu}}$

VAR. DEPENDIENTES: $u, \dfrac{v}{\sqrt{\nu}}$

PARÁMETROS: $\dfrac{I}{\sqrt{\nu}}$

$$\longrightarrow\quad u = u\left(x, \frac{y}{\sqrt{\nu}}, \frac{I}{\sqrt{\nu}}\right)\ ; \quad \frac{v}{\sqrt{\nu}} = \frac{v}{\sqrt{\nu}}\left(x, \frac{y}{\sqrt{\nu}}, \frac{I}{\sqrt{\nu}}\right)$$

Ecuaciones de dimensiones:

$$[x] = L$$

$$\left[\frac{y}{\sqrt{\nu}}\right] = \frac{L}{(L^2T^{-1})^{1/2}} = T^{1/2}$$

$$[u] = LT^{-1}$$

$$\left[\frac{v}{\sqrt{\nu}}\right] = \frac{LT^{-1}}{(L^2T^{-1})^{1/2}} = T^{-1/2}$$

$$\left[\frac{I}{\sqrt{\nu}}\right] = \frac{(LT^{-1})^2 L}{(L^2T^{-1})^{1/2}} = L^2 T^{-3/2}$$

$\longrightarrow$ Hay 2 magnitudes dimensionalmente independientes $\longrightarrow$

ADIMENSIONALIZAMOS CON $\dfrac{I}{\sqrt{\nu}}$; x

ÚNICO PARÁMETRO

OBLIGA A COGER UNA V.I.

Introducimos la función de corriente ψ:

$$u = \frac{\partial \psi}{\partial y} \longrightarrow u = \frac{\partial\left(\frac{\psi}{\sqrt{\nu}}\right)}{\partial\left(\frac{y}{\sqrt{\nu}}\right)} \qquad\qquad v = -\frac{\partial \psi}{\partial x} \longrightarrow \frac{v}{\sqrt{\nu}} = -\frac{\partial\left(\frac{\psi}{\sqrt{\nu}}\right)}{\partial x}$$

De acuerdo con las ecuaciones y condiciones de contorno, la solución es de la forma:

$$\frac{\psi}{\sqrt{\nu}} = F\left(x, \frac{y}{\sqrt{\nu}}, \frac{I}{\sqrt{\nu}}\right) \qquad\qquad \left[\frac{\psi}{\sqrt{\nu}}\right] = \frac{LT^{-1}L}{\left(L^2T^{-1}\right)^{1/2}} = LT^{-1/2}$$

Adimensionalizamos $\frac{y}{\sqrt{\nu}}$ y $\frac{\psi}{\sqrt{\nu}}$:

$$\frac{\frac{y}{\sqrt{\nu}}}{x^{\alpha}\left(\frac{I}{\sqrt{\nu}}\right)^{\beta}} \rightarrow \frac{T^{1/2}}{L^{\alpha}L^{2\beta}T^{-3\beta/2}} \rightarrow \begin{cases} \frac{1}{2} = -\frac{3\beta}{2} \\ \alpha + 2\beta = 0 \end{cases} \rightarrow \begin{cases} \alpha = \frac{2}{3} \\ \beta = -\frac{1}{3} \end{cases} \rightarrow \frac{\frac{y}{\sqrt{\nu}}}{x^{2/3}}\left(\frac{I}{\sqrt{\nu}}\right)^{1/3}$$

$$\frac{\frac{\psi}{\sqrt{\nu}}}{x^{\alpha}\left(\frac{I}{\sqrt{\nu}}\right)^{\beta}} \rightarrow \frac{LT^{-1/2}}{L^{\alpha}L^{2\beta}T^{-3\beta/2}} \rightarrow \begin{cases} -\frac{1}{2} = -\frac{3\beta}{2} \\ \alpha + 2\beta = 1 \end{cases} \rightarrow \begin{cases} \alpha = \frac{1}{3} \\ \beta = \frac{1}{3} \end{cases} \rightarrow \frac{\frac{\psi}{\sqrt{\nu}}}{\left(\frac{I}{\sqrt{\nu}}x\right)^{1/3}}$$

Por tanto:

$$\frac{\frac{\psi}{\sqrt{\nu}}}{\left(\frac{I}{\sqrt{\nu}}x\right)^{1/3}} = f\left[\frac{\frac{y}{\sqrt{\nu}}}{x^{2/3}}\left(\frac{I}{\sqrt{\nu}}\right)^{1/3}\right]$$

$$\boxed{\frac{\psi}{(I\nu x)^{1/3}} = f(\eta) \quad ; \quad \eta = \frac{y\, I^{1/3}}{3(\nu x)^{2/3}} \quad \text{VARIABLE DE SEMEJANZA}}$$

POR CONVENIENCIA, PUES SIMPLIFICARÁ LA EDO.

Hay que comprobar que la estructura de semejanza sea compatible con la ECdM$_x$ (pues la ecuación de continuidad se satisface automáticamente al introducir la función de corriente), lo que quiere decir que la ECdM$_x$ debe pasar de ser EDP(x,y) a ser EDO(η).

$$\psi = (I\nu x)^{1/3} f(\eta) \quad ; \quad \boxed{\eta = \frac{y\, I^{1/3}}{3(\nu x)^{2/3}}} \longrightarrow \boxed{\frac{\partial \eta}{\partial x} = -\frac{2}{3}\frac{\eta}{x}} \qquad \boxed{\frac{\partial \eta}{\partial y} = \frac{I^{1/3}}{3(\nu x)^{2/3}}}$$

Sustituimos en la ECdM$_x$:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$\boxed{u}$

$$u = \frac{\partial \psi}{\partial y} = \frac{d\psi}{d\eta}\frac{\partial \eta}{\partial y} = (I\nu x)^{1/3} \underbrace{\frac{df}{d\eta}}_{f'} \frac{I^{1/3}}{3(\nu x)^{2/3}} \longrightarrow \boxed{u = \frac{I^{2/3}}{3(\nu x)^{1/3}} f'}$$

$\boxed{v}$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{1}{3}(I\nu)^{1/3} x^{-2/3} f - \frac{d\psi}{d\eta}\frac{\partial \eta}{\partial x} = -\frac{1}{3}\frac{(I\nu)^{1/3}}{x^{2/3}} f - (I\nu x)^{1/3}\frac{df}{d\eta}\frac{\partial \eta}{\partial x} =$$

$$= -\frac{1}{3}\frac{(I\nu)^{1/3}}{x^{2/3}} f - (I\nu x)^{1/3} f'\left(-\frac{2}{3}\frac{\eta}{x}\right) = -\frac{1}{3}\frac{(I\nu)^{1/3}}{x^{2/3}} f + \frac{2}{3}\frac{(I\nu)^{1/3}}{x^{2/3}}\eta f' \longrightarrow$$

$$\longrightarrow \boxed{v = -\frac{(I\nu)^{1/3}}{3x^{2/3}}\left\{ f - 2\eta f' \right\}}$$

$\boxed{\frac{\partial u}{\partial x}}$

$$\frac{\partial u}{\partial x} = \frac{I^{2/3}}{3\nu^{1/3}}\frac{\partial}{\partial x}\left(x^{-1/3} f'\right) = \frac{I^{2/3}}{3\nu^{1/3}}\left[-\frac{1}{3}x^{-4/3} f' + x^{-1/3} f''\left(-\frac{2}{3}\frac{\eta}{x}\right)\right] \longrightarrow$$

$$\longrightarrow \boxed{\frac{\partial u}{\partial x} = \frac{I^{2/3}}{9\nu^{1/3} x^{4/3}}\left\{ f' + 2\eta f'' \right\}}$$

$\boxed{\frac{\partial u}{\partial y}}$

$$\frac{\partial u}{\partial y} = \frac{I^{2/3}}{3(\nu x)^{1/3}}\frac{I^{1/3}}{3(\nu x)^{2/3}} f'' \longrightarrow \boxed{\frac{\partial u}{\partial y} = \frac{I}{9\nu x} f''}$$

$\boxed{\frac{\partial^2 u}{\partial y^2}}$

$$\frac{\partial^2 u}{\partial y^2} = \frac{I}{9\nu x}\frac{I^{1/3}}{3(\nu x)^{2/3}} f''' \longrightarrow \boxed{\frac{\partial^2 u}{\partial y^2} = \frac{I^{4/3}}{27(\nu x)^{5/3}} f'''}$$

Sustituyendo:

$$-\frac{I^{2/3}}{3(\nu x)^{1/3}}\, f'\, \frac{I^{2/3}}{9\,\nu^{1/3}x^{4/3}}\left\{f' + 2\eta f''\right\} - \frac{(I\nu)^{1/3}}{3\,x^{2/3}}\left\{f - 2\eta f'\right\}\frac{I}{9\nu x}\, f'' = \nu\, \frac{I^{4/3}}{27\,(\nu x)^{5/3}}\, f'''$$

$$-\cancel{\frac{I^{4/3}}{27\,\nu^{2/3}x^{5/3}}}\,(f')^2 - \cancel{\frac{I^{4/3}}{27\,\nu^{2/3}x^{5/3}}}\,2\eta f' f'' - \cancel{\frac{I^{4/3}}{27\,\nu^{2/3}x^{5/3}}}\, f f'' + \cancel{\frac{I^{4/3}}{27\,\nu^{2/3}x^{5/3}}}\,2\eta f' f'' = \cancel{\frac{I^{4/3}}{27\,\nu^{2/3}x^{5/3}}}\, f'''$$

$$-(f')^2 - \cancel{2\eta f' f''} - f f'' + \cancel{2\eta f' f''} = f'''$$

$$\boxed{f''' + f f'' + (f')^2 = 0}$$

EDO 3^{er} ORDEN
(3 c.c. en η)

En cuanto a la compatibilidad de las condiciones de contorno:

$$y = 0 : \begin{cases} v = 0 \\ \dfrac{\partial u}{\partial y} = 0 \end{cases} \longrightarrow \eta = 0 : \begin{cases} f = 0 & (1) \\ f'' = 0 & (2) \end{cases}$$

$$y \to \infty : \quad u \to 0 \longrightarrow \eta \to \infty : f' \to 0 \;\; (3)$$

Si necesitamos que $\int_0^\infty (f')\, d\eta$ esté acotada, se debe satisfacer $\eta \to \infty : f' \to 0$, colapsando las condiciones (3) y (4)

$$\forall x : \int_0^\infty u^2\, dy = I \longrightarrow \int_0^\infty \frac{I^{4/3}}{9(\nu x)^{2/3}} (f')^2\, \frac{3(\nu x)^{2/3}}{I^{1/3}}\, d\eta = I \longrightarrow \int_0^\infty (f')^2\, d\eta = 3 \;\; (4)$$

Esta c.c. deshace la homogeneidad del problema

Por tanto, el problema queda:

$f''' + f f'' + (f')^2 = 0$
$\eta = 0 : \; f = 0 \; ; \; f'' = 0$ $\forall x : \int_0^\infty (f')^2\, d\eta = 3$

$$\xrightarrow[\text{PROBLEMA DE VALOR INICIAL}]{\text{LO CONVERTIMOS EN UN}}$$

$f''' + f f'' + (f')^2 = 0$
$\eta = 0 : \begin{cases} f = 0 \\ f' = f_0' \;/\; \int_0^\infty (f')^2\, d\eta = 3 \\ f'' = 0 \end{cases}$

Hacemos una primera integral :

$$f''' + ff'' + (f')^2 = 0$$

$$f''' + (ff')' = 0$$

$$\int f''' d\eta + \int (ff')' d\eta = 0$$

$$f'' + ff' = C_1$$

$$\Big\downarrow \; \eta = 0 : \; f = f'' = 0$$

$$C_1 = 0$$

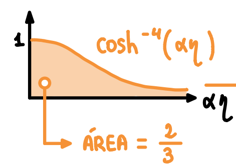
$$\longrightarrow \boxed{\begin{array}{c} f'' + ff' = 0 \\ \eta = 0 : \; f' = f'_0 \; / \; \int_0^\infty (f')^2 d\eta = 3 \end{array}}$$

Integrando de nuevo :

$$f = 2\alpha \tanh(\alpha\eta) \qquad f' = \frac{2\alpha^2}{\cosh^2(\alpha\eta)} \qquad f'' = -\frac{4\alpha^3 \operatorname{senh}(\alpha\eta)}{\cosh^3(\alpha\eta)}$$

El valor de "α" se obtiene de aplicar la condición de contorno que falta :

$$\int_0^\infty (f')^2 d\eta = 3 \longrightarrow \frac{4\alpha^4}{\alpha} \underbrace{\int_0^\infty \frac{d(\alpha\eta)}{\cosh^4(\alpha\eta)}}_{\frac{2}{3}} = 3 \longrightarrow \frac{8}{3}\alpha^3 = 3 \longrightarrow \boxed{\alpha = \frac{9^{1/3}}{2}}$$



$\eta \to \infty : f' = 0$

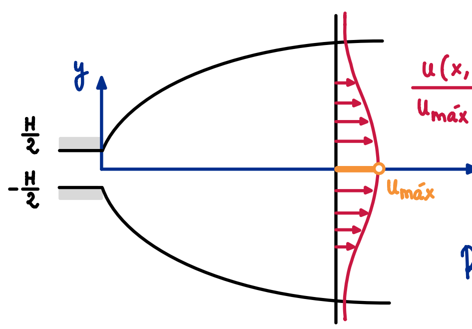
Por tanto :

$$f = 9^{1/3} \tanh\left(\frac{9^{1/3}}{2}\eta\right)$$

$$f' = \frac{9^{2/3}}{2} \operatorname{sech}^2\left(\frac{9^{1/3}}{2}\eta\right) \xrightarrow{u = \frac{I^{2/3}}{3(\nu x)^{1/3}} f'} \boxed{u = \frac{9^{2/3}}{6} \frac{I^{2/3}}{(\nu x)^{1/3}} \operatorname{sech}^2\left(\frac{9^{1/3}}{2}\eta\right)}$$

Velocidad en el centro del chorro :

$$\boxed{u_{máx} = u(\eta = 0) = \frac{(9I)^{2/3}}{6(\nu x)^{1/3}}}$$



$$\frac{u(x,y)}{u_{máx}(x)} = \operatorname{sech}^2\left(\frac{9^{1/3}}{2}\eta\right)$$

Recordamos : $\boxed{I = \frac{U^2 H}{2}}$

Flujo a través de una sección "x" del chorro:

$$q = \int_{x=cte} \vec{v}\cdot\underbrace{\vec{n}}_{\vec{i}}\, dA = 2\int_0^\infty u\,dy = \frac{6(\nu x)^{2/3}}{I^{1/3}}\int_0^\infty u\,d\eta = \frac{6(\nu x)^{2/3}}{I^{1/3}}\,\frac{(9I)^{2/3}}{3(\nu x)^{1/3}}\int_0^\infty \operatorname{sech}^2\left(\frac{9^{1/3}}{2}\eta\right)d\eta =$$

$$= 2\cdot 9^{1/3}(\nu I x)^{1/3}\tanh\left(\frac{9^{1/3}}{2}\eta\right)\Bigg|_0^\infty = 2\cdot(9\nu I x)^{1/3}\cdot(1-0) = 2\cdot(9\nu I x)^{1/3}$$

$$\boxed{q = 2\cdot(9\nu I x)^{1/3} \approx 4'1602\,(\nu I x)^{1/3}}$$

Dimensiones:

$$\left.\begin{aligned}&[q] = [u][dy] = LT^{-1}L = L^2T^{-1}\\ &([\nu][I][x])^{1/3} = \left(L^2T^{-1}(LT^{-1})^2 L L\right)^{1/3} = \left(L^6T^{-3}\right)^{1/3} = L^2T^{-1}\end{aligned}\right\}\ \checkmark\checkmark$$

Flujo en $x = 0$:

$$\boxed{q_0 = q(x=0) = \int_{-\infty}^{\infty} u_0(y)\,dy = UH}$$

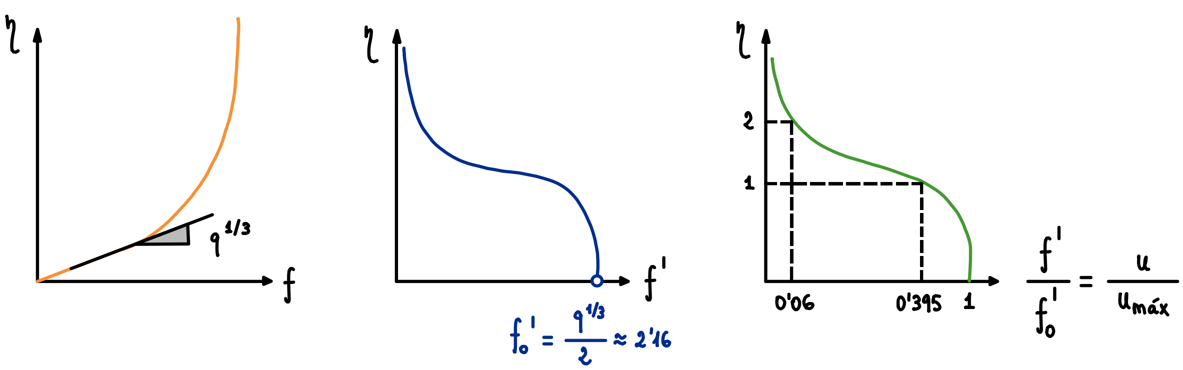
NO ES NULO OBVIAMENTE

No sustituir $x=0$ en la expresión de $q(x)$ directamente pues la expresión de "u" utilizada sólo vale lejos

Entonces:

$$\frac{q}{q_0} = \frac{2\cdot 9^{1/3}\cdot\left(\frac{U^2H}{2}\nu x\right)^{1/3}}{UH} = \left(\frac{36}{Re}\,\frac{x}{H}\right)^{1/3}$$

EL CHORRO ARRASTRA FLUIDO DESDE EL EXTERIOR Y LO INCORPORA A SU REGIÓN (ARRASTRE DEL CHORRO).



$$\underset{\substack{\uparrow\\ \text{por ejemplo}\\ \eta\sim O(1)}}{\eta = 1} \approx \frac{y}{\delta(x)} \longrightarrow \delta(x) \sim \frac{3(\nu x)^{2/3}}{I^{1/3}} \longrightarrow Re(\delta) \sim \frac{\delta(x)\,u_{máx}}{\nu} \sim x^{1/3}$$

$\uparrow x \Rightarrow \uparrow Re$ EL CHORRO ACABARÁ SIENDO TURBULENTO